The Limiting Behavior of Selected Functions

In astrophysics, many different kinds of formulae are derived for the interaction of radiation with matter. Often, astrophysicists want to know the 'limiting behavior' of the equations for extreme conditions. Here are a few examples.

$$\sigma = \frac{3}{4}s\left(\frac{1+x}{x^2}\left[\frac{2(1+x)}{1+2x} - \frac{1}{x}\ln(1+2x)\right] + \frac{1}{2x}\ln(1+2x) - \frac{1+3x}{(1+2x)^2}\right)$$

Problem 1 : The equation above is the Klien and Nishina formula for the interaction of a high-energy photon with an electron. X is the ratio of the energy carried by the photon (E = hv) compared to the rest mass energy of the electron (E = mc^2). What is the form of this equation in the limit for large X?

$$S^2 = \left[1 - \frac{2GM}{Rc^2}\right]c^2T^2 - \frac{X^2}{\left[1 - \frac{2GM}{Rc^2}\right]}$$

Problem 2 : The equation above is the Schwarschild formula for the geometric properties of the space surrounding a black hole. The mass of the black hole is given by M; displacements in time are given by T, displacements in space are given by X and the total 'space-time' displacement is given by S. The distance from the center of the black hole is given by R. Note that the first term involving time displacements T^2 has a positive sign and is called the time-like part of S^2. The second term, which involves X^2 and is called the space-like part, has a negative sign. The space-time 'signature' of S^2 is said to be (+,-) for this reason.

A) In the limit where R is very large, what is the limiting form of Schwarschild's Formula? What is this the equation for in terms of geometric figures?

B) Where does S^2 become infinite for a non-zero value for R (called the Event Horizon or R_H)?

C) What happens to the signature of S^2 as we pass from $R > R_H$ to $R < R_H$?

D) What happens at the center of the black hole as R approaches 0?

<u>Problem 1</u> – The formula can be simplified by noticing that as X becomes very large compared to 1, (1 + x) becomes x, and (1 + 2x) becomes 2x, (1 + 3x) becomes 3x, and the formula simplifies to

$$\sigma = 3/4\ s\ (1/x)[\ 2x/2x - (1/x)\ln(2x)] + (1/2x)\ln(2x) - 3x/(2x)^2).$$

This becomes $\sigma = 3/4\ s\ (1/x - (1/x^2)\ln(2x) + (1/2x)\ln(2x) - 3/4x)$

Because terms involving $1/x^2$ diminish faster than terms with 1/x, this leaves us with

$$\sigma = 3/4\ s\ [(1/2x)\ln(2x) + 1/4x]$$

which further simplifies to $\sigma = (3s/8x)[\ \ln(2x) + 1/2\ ]$

<u>Problem 2:</u> A) When R becomes very large, the equation reduces to

$$S^2 = c^2\, T^2 - X^2$$

This is the equation for a hyperbola. It shows the relationship between space and time displacements, X and T, that leads to the definition of an ‘invariant' length S. It is also the basis for the geometric interpretation of Einstein's Theory of Special Relativity.

B) S^2 becomes infinite in the space-like term when the denominator satisfies

$$0 = 1 - \frac{2G\,M}{R\,c^2} \qquad \text{Solving for R we get} \qquad R_H = \frac{2\,G\,M}{c^2}$$

This defines the radius of the event horizon for a black hole with a mass M. It can be evaluated from the constants G, c and the mass of the sun to get R_H = 2.9 kilometers for a star like our sun.

C) The sign of the time-like first term changes to negative, and the sign of the space-like second term changes to positive. The new signature for space-time becomes (-,+). This is physically interpreted to mean that the space displacement variable X has become a measure of time, and time displacement variable, T, has become a measure of space!

D) In the limit as R approaches 0, S^2 is dominated by the first term which becomes infinite, and the X^2 term becomes increasingly unimportant. This condition is called the Singularity.